Consequences of nonlocal connections in networks of chaotic maps under threshold activated coupling

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We study the effects of random nonlocal connections on networks of chaotic maps under threshold activated coupling. In threshold regimes where a large number of unsynchronized attractors occur under regular connections, we show how nonlocal rewirings yield synchronized networks. However, the dependence of the synchronized fraction on the fraction of randomized nonlocal links is typically nonmonotonic here. Further, the mean time to reach synchronization with respect to the fraction of rewiring also indicates an optimum degree of nonlocality for which synchronization is most efficiently achieved.

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I. INTRODUCTION

Lattices or networks of coupled maps were introduced as simple models capturing the essential dynamical features of extended systems [1]. Over the past decade research centered around such models has yielded suggestive conceptual frameworks for spatiotemporal phenomena in fields ranging from biology to engineering. The ubiquity of distributed complex systems has made lattices of nonlinear maps a focus of sustained research interest.

Regular networks of dynamical systems, such as with nearest-neighbor connections, have been extensively studied. But there are strong reasons to reexamine this modeling premise in the light of the fact that some degree of randomness in spatial coupling can be closer to physical reality than strict nearest-neighbor scenarios. In fact, many systems of biological, technological, and physical significance are better described by randomizing some fraction of regular links [2–8]. In particular, here we will study the spatiotemporal behavior of lattices of *threshold coupled chaotic maps*, with some of its coupling connections rewired randomly, and try to determine what dynamical properties are significantly affected by the way connections are made between the dynamical elements.

The central result of this study is that the introduction of a few random nonlocal connections yields *synchronization* in these systems. This has immediate relevance to the important problem of synchronizing spatially extended chaotic systems [9] and may have practical utility in devising synchronization methods for chaotic networks. It also provides a basis for understanding natural synchronization mechanisms in large threshold activated physical and biological systems, such as neuronal systems.

The outline of the paper is as follows: in Sec. II, we describe the details of the model, and in Sec. III we present our results on threshold coupled logistic maps. In Sec. IV we check the genericity of the results by considering lattices of circle maps under threshold activated coupling. Finally, in Sec. V we briefly summarize the principal results.

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II. MODEL

The class of systems we will focus on in this work incorporates threshold activated coupling on a lattice network of chaotic elements [10–12]. such models are relevant as prototypes for complex nonlinear self-regulatory processes in physics and biology. They yield a wealth of spatiotemporal patterns and dynamical “phases,” ranging from spatiotemporal fixed points and cycles of all orders, to scaling regimes displaying $1/f$ spectral characteristics and power law distributions.

Specifically, here we will consider a *one-dimensional* chain of chaotic maps. In these models time is discrete, labeled by n, space is discrete, labeled by i, $i=1, N$, where N is system size, and the state variable $x_n(i)$ (which in physical systems could be quantities like energy, velocity, pressure, or concentration) is continuous. Each individual site in the lattice evolves under a suitable nonlinear map $F(x)$. For instance, the local map $F(x)$ can be chosen to be the logistic map, namely

$$F(x) = rx(1-x),$$

$x \in [0,1]$, with the nonlinearity parameter r chosen in the chaotic regime, such as $r=4.0$. Now on this lattice of chaotic maps a threshold activated coupling is introduced, that is the elements couple through a threshold mechanism which is triggered when a site in the lattice exceeds the critical value x_c. Sites which exceed threshold in the lattice, namely sites i where $x_n(i) > x_c$, are known as *active* sites.

A well-studied form of threshold coupling is the nearest-neighbor interaction [10–12]. Specifically, we will consider *unidirectional* (directed) transport here. In this case, the active sites relax (or *topple*) by transporting its *excess* $\delta x = x_n(i) - x_c$ to their nearest neighbor as follows:

$$x_n(i) \to x_c,$$

$$x_n(i+1) \to x_n(i+1) + \delta x. \tag{1}$$

The coupling then induces a unidirectional transport down the array by initiating a domino effect. The boundary at i

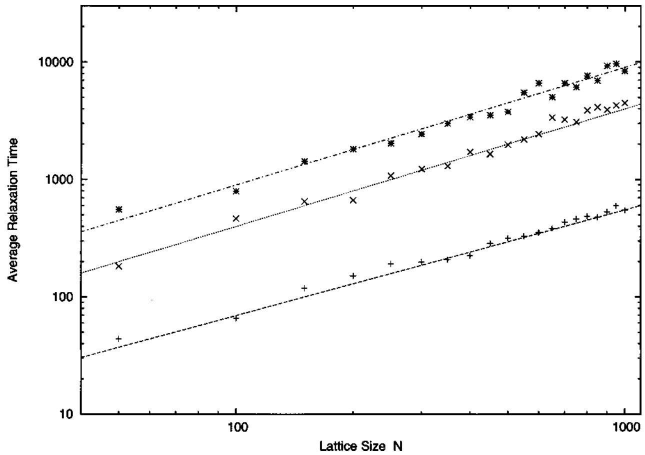

FIG. 1. Average relaxation time T_{relax} vs lattice size N, in a network of threshold coupled logistic maps. Here threshold $x_c = 0.9$ and rewiring fraction p=(a) 0.1 (+), (b) 0.5 (×), and (c) 0.9 (*). The lines display $T_{\mathrm{relax}} \sim N^{\phi}$, with $\phi = 0.9$ for the dashed line and $\phi = 1$ for the dotted and dot-dashed lines.

$=N$ is *open*. So the excess δx from the end site $i=N$ is transported out of the system.

Enough relaxation steps are allowed here in order to enable the lattice to relax fully before the next chaotic update—i.e., all $x(i) \leqslant x_c$ before the consequent chaotic update. This relaxation time T_{relax}, however grows with lattice size as a *power law*, as illustrated in Fig. 1: $T_{\mathrm{relax}} \sim N^{\phi}$, with $\phi \sim 1$. So only finite lattices have the possibility of relaxing in finite time, with finiteness of the lattice being a necessary, but not sufficient condition for complete relaxation. So all results from this work pertain to strictly finite, albeit large, lattices.

This kind of threshold activated coupling imposed on local chaos makes the above scenario especially relevant for certain mechanical systems like chains of nonlinear springs, as also for some biological systems, such as synapses of nerve tissue (note that individual neurons display complex chaotic behavior and have step-function-like responses to stimuli). The threshold mechanism is also reminiscent of the Bak-Tang-Wiesenfeld cellular automata algorithm [13], or the “sandpile” model, which gives rise to self organized criticality (SOC). The model here is, however, significantly different, the most important difference being that the threshold mechanism now occurs on a *nonlinearly evolving substrate*; i.e., there is an *intrinsic deterministic dynamics* at each site. So the local chaos here is like an *internal perturbation* as opposed to *external perturbation* in the sandpile model, which is effected by dropping “sand” from outside.

The threshold level determines the dynamical behavior of the network. Varying x_c gives rise to widely varying dynamical characteristics, ranging from spatiotemporal fixed points to multiple higher order cyclic spatiotemporal attractors. Specifically, for the case of networks of chaotic logistic maps, thresholds $x_c \leqslant 0.75$ yield fixed points with all $x_n(i) = x_c$. When $0.75 < x_c < 1.0$, the dynamics is attracted to *cycles* whose periodicities depend on the value of x_c [14]; i.e., the entire configuration $\{x(i)\}_{i=1,N}$ repeats cyclically. Depending on the initial state of the lattice, the dynamics is attracted to different cyclic configurations, and typically very many different (unsynchronized) attractors exist, in addition to the synchronized configuration [11] [see Figs. 2(a) and 3(a) for examples of generic unsynchronized attractor configurations].

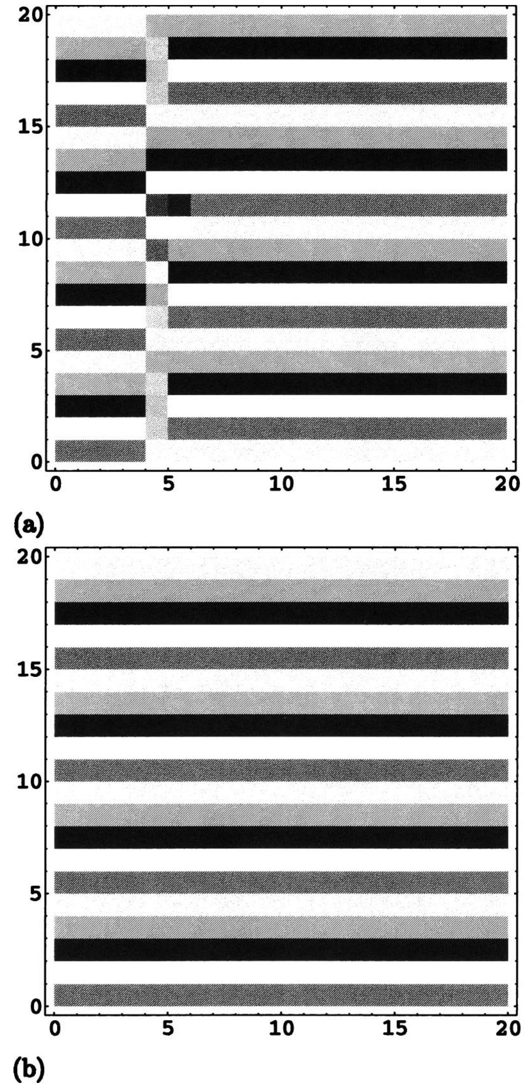

FIG. 2. Space-time density plots, for threshold coupled logistic maps. Here lattice size $N=20$, threshold $x_c = 0.94$, and rewiring fraction (a) $p=0$, (b) $p=0.1$. The lattice sites i $(i=1,\ldots,N)$ are indexed along the x axis and time n runs along the y axis. These spatial configurations are obtained from random initial conditions after transience.

Now we will consider the unidirectionally coupled system given by Eq. (1), with its coupling connections rewired randomly in varying degrees, and try to determine what dynamical properties are significantly affected by the way connections are made between elements. In our study, at every thresholding step we will transport the excess δx with probability p to randomly chosen sites in the lattice and with probability $(1-p)$ to the nearest neighbor down the array. In the sandpile analogy, strictly nearest-neighbor unidirectional coupling is the situation where a grain *always* tumbles to the next-nearest site (in the direction towards the open edge), while a random nonlocal connection is much like sand being blown to distant sites in both directions.

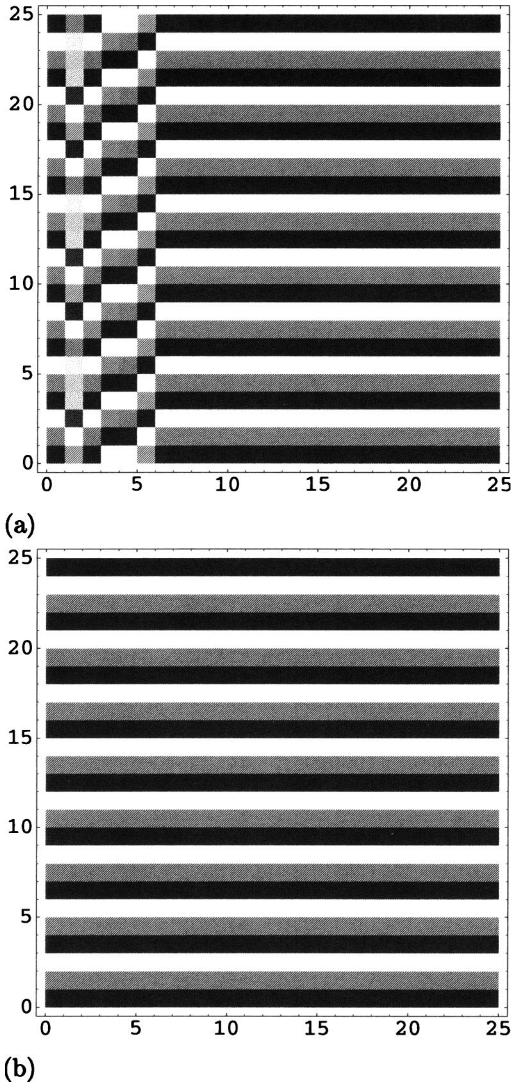

FIG. 3. Space-time density plots, for threshold coupled logistic maps. Here lattice size $N=25$, threshold $x_c=0.96$, and rewiring fraction (a) $p=0$, (b) $p=0.1$. The lattice sites i $(i=1,\ldots,N)$ are indexed along the x axis and time n runs along the y axis. These spatial configurations are obtained from random initial conditions after transience.

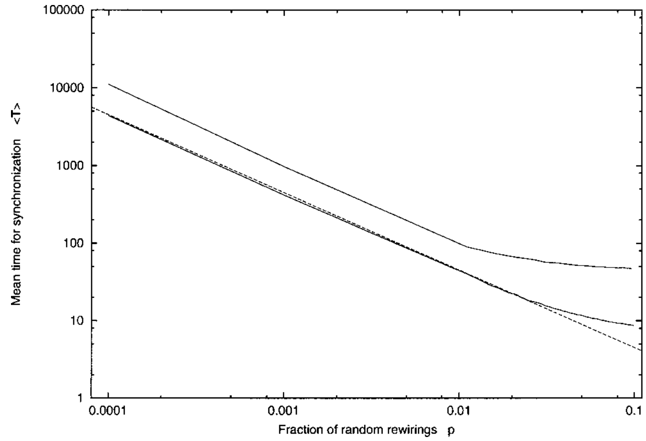

FIG. 4. Mean time $\langle T\rangle$ required to reach synchronization in threshold coupled logistic maps $(N=1000)$ vs fraction of random rewiring p, for threshold values $x_c=0.85$ (lower curve) and 0.9 (upper curve). The dashed line shows the $1/p$ line.

III. COUPLED LOGISTIC MAPS: NUMERICAL RESULTS

We will now present numerical evidence of the pronounced effect of random rewiring on synchronization in threshold coupled logistic maps. We will explore the full range of p—namely, $0\leqslant p\leqslant 1$—in different threshold regimes, in this work. Our conclusions will be based on sampling a large set of random initial conditions $(\sim 10^5)$ and lattice sizes ranging from 100 to 10 000.

When threshold x_c is small $(x_c<0.85)$ the temporal dynamics of the individual sites is either a fixed point or a 2-cycle. The simplicity of the local temporal dynamics in this threshold regime allows any generic random initial condition to reach the synchronized state—namely, the state where x at all sites are all equal—and take value x_n—i.e., $x_n(i)=x_n$ for all i, $i=1,\ldots,N$. The x_n evolve cyclically, $x_{n+k}=x_n$, with the period $k=1$ or 2 depending on threshold x_c. Further, global stability of the synchronized state is obtained for all $0\leqslant p<1$.

As the threshold x_c increases $(0.85<x_c<1)$ and the complexity of the local temporal dynamics increases, the fraction of initial conditions, f, leading to a synchronized state reduces drastically for regular coupling. For instance, for a lattice of size 1000, when $x_c=0.98$, nearest-neighbor coupling yields synchronized attractors for less than 0.01 of the 10^5 random initial conditions sampled. Figures 2(a) and 3(a) display representative unsynchronized cyclic attractors obtained from a generic random initial condition for $p=0$, for two different values of threshold.

Now when random nonlocal rewirings are introduced, that is $p>0$, there is a pronounced increase in the fraction of initial conditions attracted to the synchronized state. In fact, typically there exists some p (usually small) where the attractor for the synchronized state becomes global—i.e., the fraction of initial conditions attracted to the synchronized state $f\to 1$.

Figures 2(b) and 3(b) display representative examples of the synchronized state obtained from random initial conditions for $p=0.1$. The attracting state clearly has equal x at all sites—i.e., $x_n(i)=x_n$ for all i, $i=1\ldots,N$. The x_n evolves cyclically, $x_{n+k}=x_n$, with the period k being determined by the threshold x_c. For instance, when $x_c=0.94$, the periodicity $k=5$ (Fig. 2), and when $x_c=0.96$, the periodicity $k=4$ (Fig. 3). Note that the *only* spatial configuration that can possibly evolve in a stable cycle under random coupling is the synchronized state, where all $x_n(i)=x_n$, with x_n evolving with the periodicity dictated by the stable solutions of the single logistic map under thresholding [11]. This is unlike regular coupling where many different stable spatiotemporal cyclic configurations are possible, accounting for the coexistence of many different unsynchronized states.

Figure 4 displays the mean time $\langle T\rangle$ required to reach synchronization, with respect to the fraction of rewirings p. It is clear that up to a certain optimum value of p, $p_{\rm opt}$,

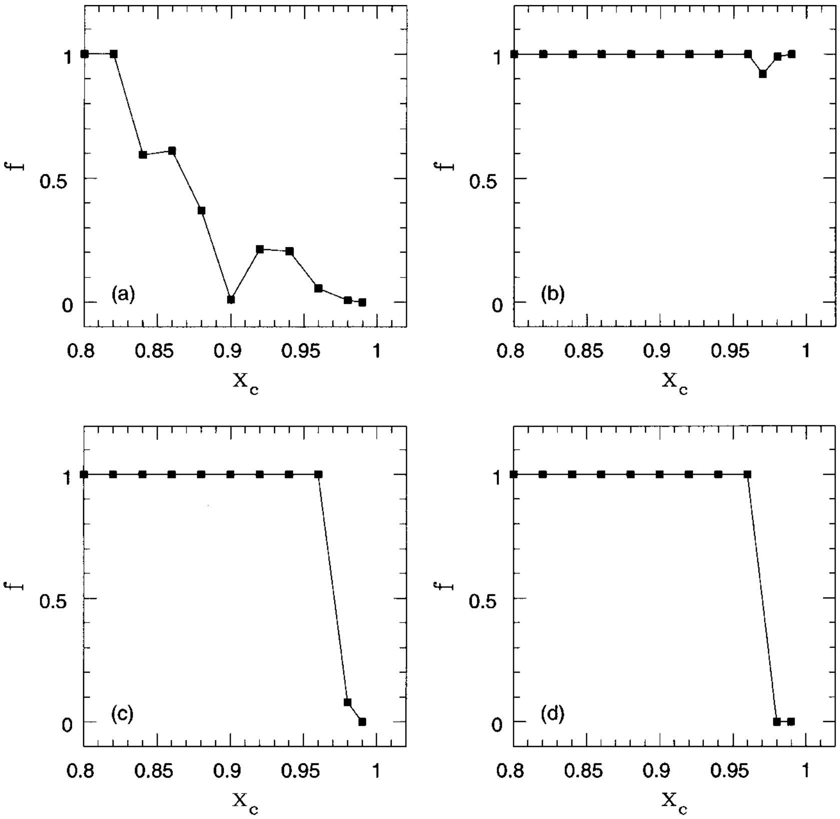

FIG. 5. Fraction of initial states leading to synchronized configurations f vs threshold x_c, for a network of size $N=1000$, and with fraction of random rewirings (a) $p=0.0$, (b) $p=0.001$, (c) $p=0.01$, and (d) $p=0.1$.

$$\langle T\rangle \sim 1/p.$$

After p_{opt}, $\langle T\rangle$ saturates or increases. The value of p_{opt} depends on the threshold x_c and is typically small. For instance, for $x_c=0.85$, $p_{\mathrm{opt}}\sim 0.1$, while for $x_c=0.9$, $p_{\mathrm{opt}}\sim 0.01$. Thus there appears to be an *optimal degree of nonlocality in coupling for which synchronization time is a minimum*. This is in contrast to random rewirings in diffusively coupled chaotic maps [15], where one obtained a monotonic dependence of various synchronization measures on p.

Note that these observations hold for synchronization within some accuracy, and the results are not significantly changed for a wide range of accuracies. Also, other statistical criteria indicative of partial synchronization—for instance, the average fraction of the lattice lying in synchronized clusters—yield similar results; namely, there is an optimal fraction of nonlocal connections for maximizing the synchronization measure.

Figure 5 displays the fraction of random initial conditions, f, which are attracted to the synchronized state, in the threshold regime $0.8<x_c<1$, under different rewiring fractions. It is evident from the figures that for nearest-neighbor interaction ($p=0$) the basin of attraction of the synchronized state is small, as all $f<0.5$. In contrast, $p>0$ support much larger basins of attraction for the synchronized state.

Interestingly, again, the synchronized fraction does not always increase with increasing random rewiring fraction. Rather, typically there exists an optimum degree of nonlocality (usually small) yielding the largest basin of attraction for the synchronized state. For instance, for $x_c=0.98$, for lattices of size $N=10\,000$ a global attractor for synchronized configurations is obtained for $p=0.001$; i.e., for very small p, a random initial condition yields a synchronized state with probability 1. In contrast, at that threshold, completely regular coupling (i.e., $p=0$) and completely random coupling (i.e., $p=1$) yield vanishingly small basins of attraction ($f\to 0$) for the synchronized state. Figure 6 illustrates this non-

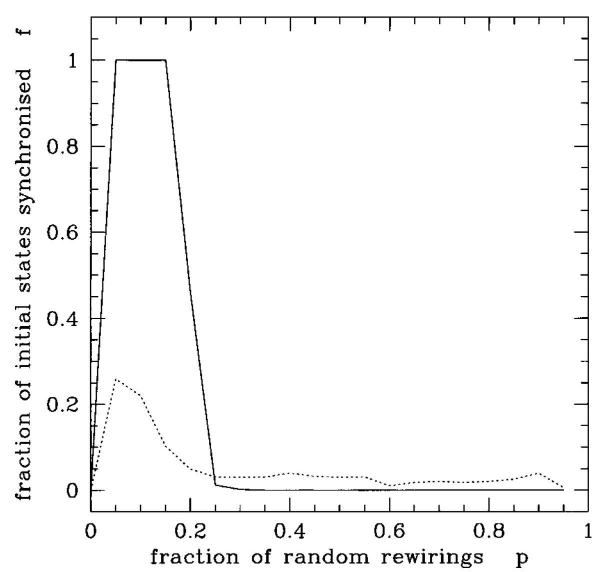

FIG. 6. Fraction of initial states leading to synchronized configurations f vs fraction of random rewirings p, in a network of threshold coupled logistic maps. Here threshold $x_c=0.97$, and the network is of size (a) $N=100$ (dashed line) and (b) $N=1000$ (solid line).

monotonicity clearly for another example: $x_c=0.97$.

So while random nonlocal rewiring does lead to better synchronization, we do not get a monotonic dependence of various synchronization properties on p, as observed in diffusively coupled chaotic networks [15]. Instead one gets a nonmonotonic variation, indicating that there exists an optimum value (or range of values) of p which yields the best synchronization results.

Note that lattice size has some effect on synchronization (see Fig. 6). Though the qualitative results appear to be quite invariant for the lattice sizes ranging from 10^2 to 10^4, the quantitative extents of the basins of attraction of the different attractors indeed changes with lattice size.

One can rationalize the synchronization brought on by random connections, and the lattice size dependence of the synchronized fraction, by noting the following: a lattice gets synchronized if all its sites exceed threshold during the threshold activated relaxation following a chaotic update. When this happens all sites will be at x_c, and their subsequent cyclic evolutions will be in exact synchrony. Second, note that sites that are "downstream" receive the cumulative excess of the "upstream" sites. So the downstream sites have a good chance of toppling one after another up to the open boundary, resulting in many contiguous synchronized sites near the edge (see for instance, in Figs. 2 and 3. This is especially true for large N, as the snowballing excess moving towards the open edge is larger in bigger lattices (as typically more active sites contribute to it). Now under random nonlocal coupling there is a finite probability that at some point in the evolution a very large amount of excess—i.e., large δx—will be transported to an undercritical upstream site during relaxation. So a random connection may take a large excess from near the open boundary and "blow it back," starting another avalanche upstream which topples many more sites as it resweeps the lattice.

However, random connections have a destabilizing effect on the relaxation process too. Relaxation is hampered as transport can now occur in both directions, while only one boundary is open. Under regular unidirectional transport the movement of excess is always directed to the open edge from where it leaves the lattice. On the other hand, random connections can result in the excess being swapped back and forth among sites in the interior of the network. So the balance between the toppling of upstream sites and the cumulative excess reaching the open edge efficiently determines the optimum p for synchronization.

IV. EFFECT OF VARYING THE LOCAL DYNAMICS

Similar results have been obtained for a threshold coupled network of circle maps [12] given by

$$F(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x). \tag{2}$$

The specific example of $\Omega=0$ and $K=4.4$ is presented here.

Figure 7 shows the fraction of random initial conditions, f, which yield the synchronized attractor with respect to threshold x_c in this system, for two values of rewiring fraction p. It is clear for $p=0$ that the basin of attraction for the

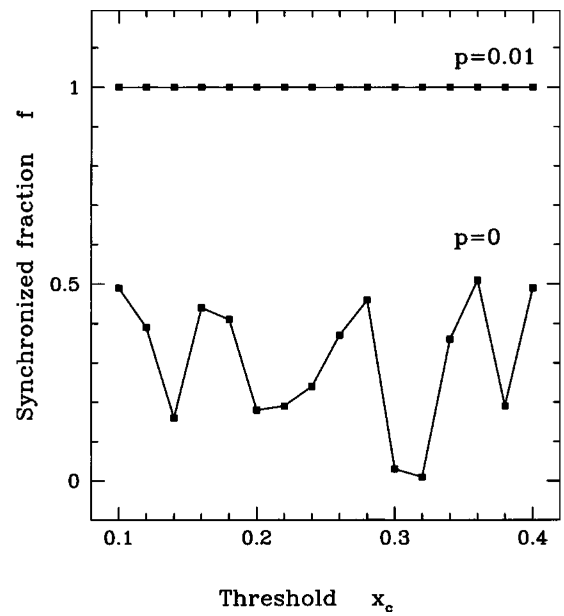

FIG. 7. Fraction of initial states leading to synchronized configurations f vs threshold x_c, in a network of threshold coupled circle maps. Here lattice size $N=1000$, with fraction of random rewiring $p=0.0$ and $p=0.01$.

synchronized state is small (as all $f<0.5$), while for $p=0.01$ we have a fully synchronized state from (nearly) all random initial conditions over a wide range of thresholds (namely all $f=1$).

Figure 8 displays the fraction of random initial conditions, f, which are attracted to the synchronized state, for threshold $x_c=0.28$, under different rewiring fractions. It is evident that for nearest-neighbor interactions $(p=0)$ the basin of attraction of the synchronized state is small $(f\sim 0.4)$, while $f=1$

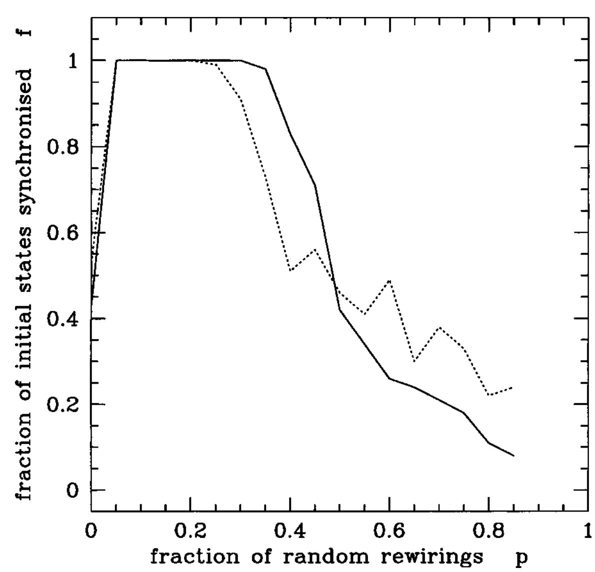

FIG. 8. Fraction of initial states leading to synchronized configurations f vs fraction of random rewiring p, in a network of threshold coupled circle maps. Here threshold $x_c=0.28$, and the network is of size (a) $N=100$ (dashed line) and (b) $N=1000$ (solid line).

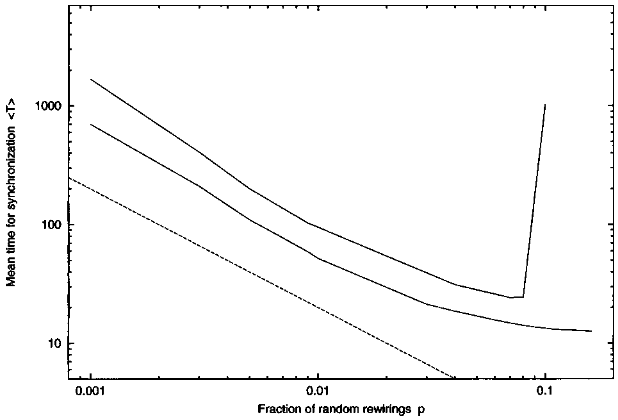

FIG. 9. Mean time $\langle T \rangle$ required to reach synchronization in threshold coupled circle maps ($N=1000$) vs fraction of random rewiring p, for threshold values $x_c=0.15$ (upper curve) and $x_c=0.35$ (lower curve). The dashed line shows the $\langle T \rangle \sim 1/p$ line.

for $0 < p < 0.4$. Interestingly, again, the dependence of the synchronized fraction is *nonmonotonic* with respect to the random rewiring fraction p, with an optimum range of randomness yielding the largest basin of attraction for the synchronized state.

Figure 9 displays the mean time $\langle T \rangle$ required to reach synchronization, with respect to the fraction of rewirings p. It is clear that up to a certain optimum value of p, $p_{\rm opt}$,

$$\langle T \rangle \sim 1/p.$$

After $p_{\rm opt}$, $\langle T \rangle$ saturates or increases. The value of $p_{\rm opt}$ depends on the threshold x_c and is typically small. For instance, for the case of $x_c=0.15$ shown in Fig. 9, $p_{\rm opt} \sim 0.1$. Thus, again, there appears to be an *optimal degree of randomness in coupling for which synchronization time is a minimum*.

V. CONCLUSIONS

In summary, then, we have shown that random rewiring of spatial connections has a pronounced effect on synchronization [16]. The robust synchronized state obtained under dilute random coupling may have significant ramifications. The key observation that the introduction of a few *nonlocal connections yields synchronization* may help devise synchronization methods for extended threshold coupled systems, as well as suggest natural synchronization mechanisms in threshold activated physical and biological systems.

ACKNOWLEDGMENT

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[16] The effect of nonlocality of connections on higher-dimensional systems is not fully clear. Preliminary examination of two-dimensional systems displays trends similar to the ones reported in this paper. However, the precise extent of these trends depends crucially on the system parameters and needs to be investigated more comprehensively.